

A LITTLE NOTE ON THE COPS & ROBBER GAME ON GRAPHS EMBEDDED IN NON-ORIENTABLE SURFACES

NANCY E. CLARKE, SAMUEL FIORINI, GWENAËL JORET, AND DIRK OLIVER THEIS

ABSTRACT. The two-player, complete information game of Cops and Robber is played on undirected finite graphs. A number of cops and one robber are positioned on vertices and take turns in sliding along edges. The cops win if, after a move, a cop and the robber are on the same vertex. The minimum number of cops needed to catch the robber on a graph is called the cop number of that graph.

Let $c(g)$ be the supremum over all cop numbers of graphs embeddable in a closed orientable surface of genus g , and likewise $\tilde{c}(g)$ for non-orientable surfaces. It is known (Andrea, 1986) that if X is a fixed surface, the maximum over all cop numbers of graphs embeddable in this surface is finite. More precisely, Quilliot (1983 probably) showed that $c(g) \leq 2g + O(1)$, and Schroeder (2001) sharpened this to $c(g) \leq \frac{3}{2}g + 3$. In his paper, Andrea gave the bound $\tilde{c}(g) \leq O(n)$ with a weak constant, and posed the question whether a stronger bound can be obtained. In a recent preprint, Nowakowski & Schroeder obtained $\tilde{cop}(g) \leq 2g + 1$.

In this short note, we show $\tilde{c}(g) = c(g - 1)$ for any $g \geq 1$. As a corollary, using Schroeder's results, we obtain the following: the maximum cop number of graphs embeddable in the projective plane is 3; the maximum cop number of graphs embeddable in the Klein Bottle is at most 4, $\tilde{c}(3) \leq 5$, and $\tilde{c}(g) \leq \frac{3}{2}g + 3/2$ for all other g .

In this paper, all graphs will be finite, undirected and simple. Let G be a graph. For an integer $k \geq 1$, the *Cops and Robber game with k cops* on G is played by two players, the cop player and the robber. The rules are as follows. First, the cop player places k cops on (not necessarily distinct) vertices of the graph; second the robber chooses a vertex. Now, starting with the cop player, the two players move alternately. A cop move consists in selecting a possibly empty subset of the cops, and placing each of the selected cops from the vertex he is standing on to a neighboring vertex. Similarly, in the robber move, the robber may move to an adjacent vertex or stay where he is. The game ends when a cop and the robber are positioned on the same vertex, i.e., the cops catch the robber, in which case the cops win. If the game proceeds forever the robber wins. Both players have complete information.

We say that G is *k -copwin*, if, in the k -cops game, the cops have a winning strategy. The smallest integer k such that a graph G is k -copwin is called its *cop number* and denoted by $c(G)$.

Date: Fri Aug 1 21:40:07 CEST 2008.

2000 Mathematics Subject Classification. 05C99, 05C10; 91A43.

Key words and phrases. Games on graphs, cops and robber game, cop number, graphs on surfaces.

This research was done during a stay of NC in Brussels.

GJ is a Research Fellow of the *Fonds National de la Recherche Scientifique (F.R.S.–FNRS)*.

DOT supported by *Fonds National de la Recherche Scientifique (F.R.S.–FNRS)*.

Nowakowski & Winkler [6] and Quilliot [7] have characterized the class of 1-copwin graphs. Families of graphs with unbounded cop number have been constructed [1], even families of d -regular graphs, for each $d \geq 3$ [2].

By a surface, we mean a closed surface, i.e., a compact two dimensional topological manifold w/o boundary. For any non-negative integer g we denote by $c(g)$ the supremum over all $c(G)$ with G ranging over all graphs embeddable in a orientable surface of genus g , and we call this the cop number of the surface. Similarly, we define the cop number $\tilde{c}(g)$ of a non-orientable surface of genus g to be the supremum over all $c(G)$ with G ranging over all graphs embeddable in this surface.

Aigner & Fromme [1] proved that the cop number of the sphere is equal to three, i.e., $c(0) = 3$. Quilliot [8] gave an inductive argument to the effect that the cop number of an orientable surface of genus g is at most $2g + 3$. Schröder [9] was able to sharpen this result to $c(g) \leq \frac{3}{2}g + 3$. He also proved that the cop number of the double torus is at most five.

Andreae [3] generalized the work of Aigner & Fromme. He proved that, for any graph H satisfying a mild connectivity assumption, the class of graphs which do not contain H as a minor has cop number bounded by a constant depending on H . Using this, and the well known formula for non-orientable genus of a complete graph, he obtained an upper bound for the cop number of a non-orientable surface of genus g , namely

$$\tilde{c}(g) \leq \left(\frac{\lfloor 7/2 + \sqrt{6g + 1/4} \rfloor}{2} \right)$$

In recent work, Nowakowski & Schröder[5] prove a much stronger bound: $\tilde{c}(g) \leq 2g + 1$. For this, they use a series of technically challenging and ingenious arguments.

In this mini note, we prove the following.

Theorem 1. $c(\lfloor g/2 \rfloor) \leq \tilde{c}(g) \leq c(g - 1)$

This immediately improves the best known upper bound for the non-orientable surface of genus g to $\tilde{c}(g) \leq \frac{3}{2}(g - 1) + 3 = \frac{3}{2}(g + 1)$, by Schröder's on the cop numbers of orientable surfaces mentioned above. The following table gives the new and status quo for the concrete upper bounds.

N/o genus	1	2	3	4	5	6	7
N. & S. [5]	3	5	7	9	11	13	15
Here	3	4	5 ¹	7	9	10	12

We say that a *weak cover* of H by G is a surjective mapping $p: V(G) \rightarrow V(H)$ which maps vertex neighborhoods onto vertex neighborhoods, i.e., for every vertex u of G , we have $p(N(u)) = N(p(u))$. (This terminology lends on the classical definition of a “cover” w/o weak, where the restriction to the vertex neighborhood $p: N(u) \rightarrow N(p(u))$ is required to be a bijection.) Using the same technique as for the inequality “ \leq ” in the proof of Theorem 1, it is possible to show the following:

Lemma 2. *If G is a weak cover of H , then $c(H) \leq c(G)$.*

¹Using Schröder's [9] result that $c(2) \leq 5$.

This is similar in spirit to the seminal result of Berarducci & Ingridila [4], saying that if H is a retract of G , then the same inequality holds. Note, however, that neither of the two notions generalizes the other. We will not prove Lemma 2; the proof is only slightly more technical than the geometric proof of Theorem 1.

PROOFS

Familiarity with the classification of combinatorial surfaces is assumed. See any textbook on topology. We will make use of the standard representation of surfaces as quotients of polygonal discs with labelled and directed edges. Each label occurs twice, and the two edges with the same label are identified according to their orientations. Reading the labels of the edges in counterclockwise (i.e., *positive*) order and adding an exponent -1 whenever the orientation of the edges is negative (i.e., clockwise) gives the *word* of the surface.

Proof that $\tilde{c}(g) \geq c(\lfloor g/2 \rfloor)$. This is the easy one, using the following well-known fact. For a graph G , let $\gamma(G)$ denote the smallest integer g such that G can be embedded in an orientable surface of genus g ; similarly define $\tilde{\gamma}(G)$ as the smallest integer g such that G can be embedded in a non-orientable surface of genus g .

Lemma 3 (Folklore). *For any graph G we have $\tilde{\gamma}(G) \leq 2\gamma(G) + 1$.*

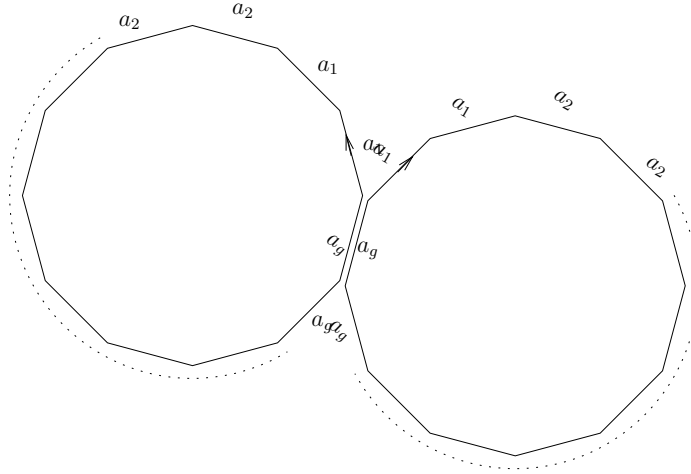
Proof. For $g := \gamma(G)$, suppose that G is embedded in the surface X with word $x_1 y_1 x_2^{-1} y_2^{-1} \dots x_g y_g x_g^{-1} y_g^{-1}$. W.l.o.g, we can assume that the point of X which is the image of all the vertices of the polygonal disc is used in the drawing of the graph. Hence, we can add a “crosscap”, i.e., two edges in positive orientation, both with label a , and embed the graph on the new surface X^+ . The standard arguments used in the classification of combinatorial surfaces now yield that X^+ is a non-orientable surface of genus $2g + 1$. \square

This lemma immediately implies $c(g) \leq \tilde{c}(2g + 1)$, and hence $\tilde{c}(g) \geq c(\lfloor g/2 \rfloor)$.

Proof that $\tilde{c}(g) \leq c(\lfloor g/2 \rfloor)$. For this inequality, we make use of the well-known fact that every manifold X has a 2-sheeted covering $X' \rightarrow X$ by an orientable manifold. If X is a non-orientable surface of genus g , it is easy to see that the standard construction (again, see a textbook on topology) yields a surface of genus $g - 1$. We give the details in the following lemma.

Lemma 4. *A non-orientable surface of genus g has an orientable surface of genus $g - 1$ as a 2-sheeted covering space.*

Proof. We prove by picture.



□

For the proof that $\tilde{c}(g) \leq c(g-1)$, let X be the non-orientable surface of genus g on which a graph G is embedded. We identify the graph G with its embedding, i.e., we think of the vertex set $V(G)$ as a set of points of X and the edge set of $E(G)$ as a set of internally disjoint injective curves connecting the respective end vertices of the edge.

By Lemma 4, there exists a covering $p: X' \rightarrow X$ of X by an orientable surface X' with genus $g' := g - 1$. Consider the graph G' whose vertex set is $\{p^{-1}(V(G))\}$ and whose edge sets consists of the curves obtained by lifting the edges of G . By construction, G' is embedded in the orientable surface Y of genus g .

We now give a strategy for $k := c(g)$ cops to win the Cops and Robber game on G , by “simulating” a game on G' and using any winning strategy for k cops on this graph, who chase an “imaginary” robber. In such a strategy, the k cops first pick their starting vertices $u_1, \dots, u_k \in V(G')$. In the strategy for G , we let the starting vertices be $p(u_1), \dots, p(u_k)$. Suppose now that, in the game in G , the robber picks a starting vertex r . We choose a starting vertex for an imaginary robber on G' arbitrarily in the fibre $p^{-1}(r)$.

Throughout the game, the position of each checker in G' will be in the fibre $p^{-1}(x)$ of the position x of the corresponding checker in G . Moreover, the movements of the checkers on G describe curves on X , which can be lifted (unique, although this is inessential) to curves on Y forming walks in G' .

Now, whenever it be the cops' turn in any game on G , the robber is at a certain vertex s of G' , and the k cops are on vertices v_1, \dots, v_k . The strategy for the cops on G' now prescribes moves for the cops. The corresponding moves in G are then given as images under p .

Since we chose a winning strategy, after a finite number of times, the imaginary robber on G' will be on the same vertex as a cop in G' . Consequently, the same holds on G , thus the cops have won the game on G .

CONCLUSION

We conclude with a conjecture.

Conjecture. $\tilde{c}(g) = c(\lfloor g/2 \rfloor)$.

One might wonder whether it is possible to improve Theorem 1 by taking a different covering, or possibly a branched covering. This is impossible: It is a well-known fact that, whenever $p: X' \rightarrow X$ is a (branched) covering with X' orientable and X non-orientable, then p lifts to a (branched) covering $\tilde{p}: X' \rightarrow \tilde{X}$, where \tilde{X} is the orientable double cover constructed in Lemma 4.

REFERENCES

- [1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Appl. Math.*, 8:1–12, 1984.
- [2] T. Andreae. Note on a pursuit game played on graphs. *Discrete Appl. Math.*, 9:111–115, 1984.
- [3] T. Andreae. On a pursuit game played on graphs for which a minor is excluded. *J. Combin. Th., Ser. B*, 41:37–47, 1986.
- [4] A. Berarducci and B. Intrigila. On the cop number of a graph. *Adv. Appl. Math.*, 14:389–403, 1993.
- [5] R. J. Nowakowski and B. S. W. Schröder. Bounding the cop number using the crosscap number. Preprint.
- [6] R. J. Nowakowski and P. Winkler. Vertex to vertex pursuit in a graph. *Discrete Math.*, 43:23–29, 1983.
- [7] A. Quilliot. *Thèse d’Etat*. PhD thesis, Université de Paris VI, 1983.
- [8] A. Quilliot. A short note about pursuit games played on a graph with a given genus. *J. Combin. Theory Ser. B*, 38:89–92, 1985.
- [9] B. S. W. Schröder. The copnumber of a graph is bounded by $\lfloor \frac{3}{2}\text{genus}(G) \rfloor + 3$. In “*Categorical Perspectives*” — *Proceedings of the Conference in Honor of George Strecker’s 60th Birthday*, pages 243–263. Birkhäuser, 2001.

NANCY E. CLARKE, ACADIA UNIVERSITY, WOLFVILLE, CA
E-mail address: nancy.clarke@acadiau.ca

SAMUEL FIORINI, SERVICE DE GÉOMETRIE COMBINATOIRE ET THÉORIE DES GROUPES,
 DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ LIBRE DE BRUXELLES, BRUSSELS, BEL-
 GIUM
E-mail address: sfiorini@ulb.ac.be

GWENAËL JORET, DÉPARTEMENT D’INFORMATIQUE, UNIVERSITÉ LIBRE DE BRUXELLES,
 BRUSSELS, BELGIUM
E-mail address: gjoret@ulb.ac.be

DIRK OLIVER THEIS, SERVICE DE GÉOMETRIE COMBINATOIRE ET THÉORIE DES GROUPES,
 DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ LIBRE DE BRUXELLES, BRUSSELS, BEL-
 GIUM
E-mail address: Dirk.Theis@ulb.ac.be